White Noise Analysis of the Frequency Lock Loop

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1 Block Diagram and Preliminaries

This appendix analyzes the white noise performance of the frequency lock loop (FLL) that uses a cross product discriminator to develop the frequency error signal. This analysis is based on the nice work of Van Dierendonck and Socci, 1982.

The block diagram of this structure is shown in Figure 12.A.1. As shown, our considerations begin after the signal has been down converted to an intermediate frequency (IF). We assume that the carrier has been wiped off and the signal from one satellite is given by

$$\sqrt{C}D(t-\tau)x(t-\tau)\exp j(2\pi\Delta f_D t+\Delta\theta)$$

The power in the signal is given by C, $D(t)$ denotes the navigation data, the code is given by $x(t)$, and the signal delay is τ. The frequency lock loop is not perfectly aligned with the incoming frequency, and this frequency offset manifests itself as a frequency error, Δf_D and carrier phase error $\Delta\theta$. We will have more to say about these error signals in what follows.

This signal is correlated with the code replica carried by the receiver. Correlation yields

$$\tilde{S}=\frac{\sqrt{C}D}{T_{CO}}\int_0^{T_{CO}} x(t-\tau)x(t-\hat{\tau})\exp j(2\pi\Delta f_D t+\Delta\theta)dt \quad (1)$$

As shown, the signal is multiplied by $x(t-\hat{\tau})$ and then integrated for T_{CO} seconds, where T_{CO} constitutes the coherent integration time. $\tilde{S}$ is the contribution of one satellite signal to the output of the complex correlator. In general, the correlator outputs also contain contributions from other satellite signals and noise. The contribution from the other satellite signals is called multiple access interference, which we will ignore. For GPS, multiple access interference is small compared to the noise contribution.

As described in Chapter 11, the real part of the complex correlation comes from the inphase correlator, and the imaginary part comes from the quadrature correlator. The prompt correlation outputs at time $t=nT_{CO}$ are given by

$$\tilde{Z}_{P,n} \quad = \quad \tilde{S}_{P,n}+\tilde{\eta}_{P,n}$$

$$
\begin{aligned}
&= S_{I,P,n} + jS_{Q,P,n} + \eta_{I,P,n} + j\eta_{Q,P,n} \\
Z_{I,P,n} &= \mathrm{Re}\left\{\tilde{Z}_{P,n}\right\} \\
&= S_{I,P,n} + \eta_{I,P,n} \\
Z_{Q,n} &= \mathrm{Im}\left\{\tilde{Z}_{P,n}\right\} \\
&= S_{Q,P,n} + \eta_{Q,P,n}
\end{aligned} \tag{2}
$$

The signal contributions to the inphase and quadrature samples are given by $S_{I,P,n}$ and $S_{Q,P,n}$, and the noise contributions are given by $\eta_{I,P,n}$ and $\eta_{Q,P,n}$. We need expressions for all four terms.

The signal contributions are developed by assuming that the prompt correlator is accurate, $\hat{\tau} - \tau \approx 0$. For convenience, we also drop the subscript P for denoting the prompt correlator, because all of our subsequent work is for the prompt correlator. In this case, we may write

$$
\begin{aligned}
\tilde{S}_n &= \frac{\sqrt{C}D}{T_{CO}} \int_0^{T_{CO}} x(t-\tau)x(t-\hat{\tau}) \exp j(2\pi\Delta f_D t + \Delta\theta_n) dt \\
&= \frac{\sqrt{C}D}{T_{CO}} \int_0^{T_{CO}} \exp j(2\pi\Delta f_D t + \Delta\theta_n) dt \\
&= \frac{\sqrt{C}D}{T_{CO}} \exp j(\Delta\theta_n) \left. \frac{\exp j(2\pi\Delta f_D t)}{j2\pi\Delta f_D} \right|_0^{T_{CO}} \\
&= \sqrt{C}De^{j\Delta\theta_n} \frac{\exp j(2\pi\Delta f_D T_{CO}) - 1}{j2\pi\Delta f_D T_{CO}} \\
&= \sqrt{C}De^{j\Delta\theta_n} \frac{\exp j(\pi\Delta f_D T_{CO})(\exp j(\pi\Delta f_D T_{CO}) - \exp -j(\pi\Delta f_D T_{CO}))}{j2\pi\Delta f_D T_{CO}} \\
&= \sqrt{C}De^{j\Delta\theta'_n} \frac{2j\sin(\pi\Delta f_D T_{CO})}{j2\pi\Delta f_D T_{CO}} \\
&= \sqrt{C}De^{j\Delta\theta'_n} \mathrm{sinc}(\pi\Delta f_D T_{CO})
\end{aligned} \tag{3}
$$

This result subsumes the phase shift $\pi\Delta f_D T_{CO}$ in $\Delta\theta'$, and from this point we simply return to the notation $\Delta\theta$ for the phase error inclusive of this additional shift.

For small values of Δf_D, we may finally write

$$
\begin{aligned}
\tilde{S}_n &= \sqrt{C}De^{j\Delta\theta_n} \\
S_{I,n} &= \sqrt{C}D\cos(\Delta\theta_n) \\
S_{Q,n} &= \sqrt{C}D\sin(\Delta\theta_n)
\end{aligned} \tag{4}
$$

The received signal has additive white noise denoted $n(t)$, and so the noise samples are given by

$$
\begin{aligned}
\eta_{I,n} &= \frac{\sqrt{2}}{T_{CO}} \int_0^{T_{CO}} n(t)x(t-\hat{\tau}) \cos(2\pi(f+\hat{f}_D)t + \hat{\theta}_n) dt \\
\eta_{Q,n} &= \frac{\sqrt{2}}{T_{CO}} \int_0^{T_{CO}} n(t)x(t-\hat{\tau}) \sin(2\pi(f+\hat{f}_D)t + \hat{\theta}_n) dt
\end{aligned} \tag{5}
$$

These noise samples have zero mean.

$$\begin{aligned}
E\{\eta_{I,n}\} &= E\{\eta_{Q,n}\} \\
&= E\left\{\frac{\sqrt{2}}{T_{CO}}\int_0^{T_{CO}} n(t)x(t-\hat{\tau})\cos(2\pi(f+\hat{f}_D)t+\hat{\theta})dt\right\} \\
&= \frac{\sqrt{2}}{T_{CO}}\int_0^{T_{CO}} E\{n(t)\}x(t-\hat{\tau})\cos(2\pi(f+\hat{f}_D)t+\hat{\theta})dt \\
&= 0
\end{aligned} \tag{6}$$

The variances are a little more challenging. Since $E\{\eta_{I,n}\} = 0$, we may write

$$\begin{aligned}
\mathrm{var}\{\eta_{I,n}\} &= E\{(\eta_{I,n} - E\{\eta_{I,n}\})^2\} \\
&= E\{\eta_{I,n}^2\} \\
&= E\left\{\frac{\sqrt{2}}{T_{CO}}\int_0^{T_{CO}} n(t)x(t-\hat{\tau})\cos(2\pi(f+\hat{f}_D)t+\hat{\theta})dt\right. \\
&\times \left.\frac{\sqrt{2}}{T_{CO}}\int_0^{T_{CO}} n(s)x(s-\hat{\tau})\cos(2\pi(f+\hat{f}_D)s+\hat{\theta})ds\right\} \\
&= \frac{2}{T_{CO}^2}\int_0^{T_{CO}}\int_0^{T_{CO}} E\{n(t)n(s)\}x(t-\hat{\tau})x(s-\hat{\tau}) \\
&\times \cos(2\pi(f+\hat{f}_D)t+\hat{\theta})\cos(2\pi(f+\hat{f}_D)s+\hat{\theta})dtds
\end{aligned} \tag{7}$$

Since the carrier frequency, f, varies much more quickly then the code, $x(t)$, the double frequency term will average to zero, and we may write

$$\begin{aligned}
\mathrm{var}\{\eta_{I,n}\} &= \frac{1}{T_{CO}^2}\int_0^{T_{CO}}\int_0^{T_{CO}} E\{n(t)n(s)\}x(t-\hat{\tau})x(s-\hat{\tau})dtds \\
&= \frac{1}{T_{CO}^2}\int_0^{T_{CO}}\int_0^{T_{CO}} \frac{N_0}{2}\delta(t-s)x(t-\hat{\tau})x(s-\hat{\tau})dtds \\
&= \frac{N_0}{2T_{CO}^2}\int_0^{T_{CO}} x(t-\hat{\tau})x(t-\hat{\tau})dt \\
&= \frac{N_0}{2T_{CO}}
\end{aligned} \tag{8}$$

This result is the same as we found in Chapter 10 for the baseband analysis and Chapter 11 for the RF signal acquisition analysis. In all three cases, noise decreases with increasing averaging time - a sensible result.

With a similar proof, we find that the variance of $\eta_{Q,n}$ is also equal to $N_0/2T_{CO}$. Consequently, we write

$$\sigma^2 = \mathrm{var}\{\eta_{I,n}\} = \mathrm{var}\{\eta_{Q,n}\} = \frac{N_0}{2T_{CO}} \tag{9}$$

The samples, given by $Z_{I,n} = S_{I,n} + \eta_{I,n}$ and $Z_{Q,n} = S_{Q,n} + \eta_{Q,n}$ are the inputs to the cross product discriminator.

2 Cross Product Discriminator

In what follows, we seek an error signal, or discriminator, that is linear, or approximately linear in the frequency error, Δf_D. For this analysis, we assume that the frequency error is small enough that $\text{sinc}(\pi\Delta f_D T_{CO}) \approx 1$. For the coherent integration times used in GPS receivers, this means that $\Delta f_D < 200$ Hz. For errors of this size, the phase term continues to be influenced by the frequency error and we write $\Delta\theta_n = \Delta\theta_0 + 2\pi n\Delta f_D T_{CO}$. In this expression, the phase error is equal to some initial phase error plus the frequency error multiplied by the lapsed time. This means

$$\begin{aligned} S_{I,n} &= \sqrt{C}D\cos(\Delta\theta_n) \\ &= \sqrt{C}D\cos(\Delta\theta_0 + 2\pi n\Delta f_D T_{CO}) \\ S_{Q,n} &= \sqrt{C}D\sin(\Delta\theta_n) \\ &= \sqrt{C}D\sin(\Delta\theta_0 + 2\pi n\Delta f_D T_{CO}) \end{aligned} \tag{10}$$

The cross product discriminator implements the following trigonometric identity.

$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta) \approx \alpha - \beta \tag{11}$$

This action exposes the frequency error of interest. More completely, we may write

$$\begin{aligned} S_{I,n-1}S_{Q,n} - S_{I,n}S_{Q,n-1} &= C(\cos(\Delta\theta_{n-1})\sin(\Delta\theta_n) - \cos(\Delta\theta_n)\sin(\Delta\theta_{n-1})) \\ &= C\sin(\Delta\theta_n - \Delta\theta_{n-1}) \\ &= C\sin(2\pi\Delta f_D T_{CO}) \\ &\approx C2\pi\Delta f_D T_{CO} \end{aligned} \tag{12}$$

As shown in Figure 12.A.1, we average N of these error signals together to yield our final error signal.

$$L_f = \sum_{n=2}^{N} S_{I,n-1}S_{Q,n} - S_{I,n}S_{Q,n-1} \tag{13}$$

The sensibility of this discriminator can be seen by considering the result in the absence of noise.

$$\sum_{n=2}^{N} S_{I,n-1}S_{Q,n} - S_{I,n}S_{Q,n-1} \approx 2\pi C(N-1)\Delta f_D T_{CO}$$

The sum over N constitutes a non-coherent sum. In the presence of noise, this averaging provides a performance improvement, but this non-coherent integration gain is not as strong as the coherent integration gain, T_{CO}.

The expected value of the discriminator contains the signal contributions.

$$\begin{aligned} E\{L_f\} &= \sum_{n=2}^{N} E\{Z_{I,n-1}Z_{Q,n} - Z_{I,n}Z_{Q,n-1}\} \\ &= \sum_{n=2}^{N} E\{(S_{I,n-1}+\eta_{I,n-1})(S_{Q,n}+\eta_{Q,n}) - (S_{I,n}+\eta_{I,n})(S_{Q,n-1}+\eta_{Q,n-1})\} \\ &= \sum_{n=2}^{N} E\{S_{I,n-1}S_{Q,n} + S_{I,n-1}\eta_{Q,n} + S_{Q,n}\eta_{I,n-1} + \eta_{I,n-1}\eta_{Q,n} \end{aligned}$$

$$
\begin{aligned}
&- \quad S_{I,n}S_{Q,n-1} - S_{I,n}\eta_{Q,n-1} - S_{Q,n-1}\eta_{I,n} - \eta_{I,n}\eta_{Q,n-1}\} \\
&= \quad \sum_{n=2}^{N} S_{I,n-1}S_{Q,n} - S_{I,n}S_{Q,n-1} \\
&= \quad 2\pi(N-1)CT_{CO}\Delta f_D
\end{aligned} \tag{14}
$$

Later, we will need to identify this quantity squared.

$$
\begin{aligned}
(E\{L_f\})^2 &= \sum_{n=2}^{N}\sum_{n'=2}^{N}(S_{I,n-1}S_{Q,n} - S_{I,n}S_{Q,n-1})(S_{I,n'-1}S_{Q,n'} - S_{I,n'}S_{Q,n'-1}) \\
&= \sum_{n=2}^{N}\sum_{n'=2}^{N} S_{I,n-1}S_{I,n'-1}S_{Q,n}S_{Q,n'} - S_{I,n-1}S_{I,n'}S_{Q,n}S_{Q,n'-1} \\
&- \quad S_{I,n}S_{I,n'-1}S_{Q,n-1}S_{Q,n'} + S_{I,n}S_{I,n'}S_{Q,n-1}S_{Q,n'-1}
\end{aligned} \tag{15}
$$

3 Variance of the Error Signal

Now we begin the hardest task - finding the variance of the error signal. Our approach is based on the following well known relationship between the variance and the first two moments.

$$
\mathrm{var}(X) = E\{X^2\} - (E\{X\})^2
$$

Since we already have $(E\{X\})^2$ from the last subsection, we pursue $E\{X^2\}$.

$$
\begin{aligned}
E\{L_f^2\} &= E\{(\sum_{n=2}^{N} Z_{I,n-1}Z_{Q,n} - Z_{I,n}Z_{Q,n-1})^2\} \\
&= E\{\sum_{n=2}^{N} Z_{I,n-1}Z_{Q,n} - Z_{I,n}Z_{Q,n-1} \sum_{n'=2}^{N} Z_{I,n'-1}Z_{Q,n'} - Z_{I,n'}Z_{Q,n'-1}\} \\
&= \sum_{n=2}^{N}\sum_{n'=2}^{N} E\{Z_{I,n-1}Z_{Q,n}Z_{I,n'-1}Z_{Q,n'}\} - E\{Z_{I,n-1}Z_{Q,n}Z_{I,n'}Z_{Q,n'-1}\} \\
&- \quad E\{Z_{I,n}Z_{Q,n-1}Z_{I,n'-1}Z_{Q,n'}\} + E\{Z_{I,n}Z_{Q,n-1}Z_{I,n'}Z_{Q,n'-1}\}
\end{aligned} \tag{16}
$$

In what follows, these last four products are called terms one through four.

Term one is developed as follows.

$$
\begin{aligned}
E\{Z_{I,n-1}Z_{Q,n}Z_{I,n'-1}Z_{Q,n'}\} &= E\{(S_{I,n-1}+\eta_{I,n-1})(S_{Q,n}+\eta_{Q,n})(S_{I,n'-1}+\eta_{I,n'-1})(S_{Q,n'}+\eta_{Q,n'})\} \\
&= E\{(S_{I,n-1}S_{Q,n} + S_{I,n-1}\eta_{Q,n} + S_{Q,n}\eta_{I,n-1} + \eta_{I,n-1}\eta_{Q,n}) \\
&\quad (S_{I,n'-1}S_{Q,n'} + S_{I,n'-1}\eta_{Q,n'} + S_{Q,n'}\eta_{I,n'-1} + \eta_{I,n'-1}\eta_{Q,n'})\} \\
&= E\{S_{I,n-1}S_{Q,n}S_{I,n'-1}S_{Q,n'} + S_{I,n-1}S_{I,n'-1}\eta_{Q,n}\eta_{Q,n'} \\
&+ \quad S_{Q,n}S_{Q,n'}\eta_{I,n-1}\eta_{I,n'-1} + \eta_{I,n-1}\eta_{I,n'-1}\eta_{Q,n}\eta_{Q,n'}\} \\
&= S_{I,n-1}S_{Q,n}S_{I,n'-1}S_{Q,n'} + S_{I,n-1}S_{I,n'-1}\sigma^2\delta_{n,n'} \\
&+ \quad S_{Q,n}S_{Q,n'}\sigma^2\delta_{n-1,n'-1} + \sigma^4\delta_{n-1,n'-1}\delta_{n,n'}
\end{aligned} \tag{17}
$$

This equation uses the Kronecker delta function to distinguish noise samples that are correlated from those that are not correlated.

$$\delta_{i,k} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Term two is given by

$$\begin{aligned}
E\{Z_{I,n-1}Z_{Q,n}Z_{I,n'}Z_{Q,n'-1}\} &= E\{(S_{I,n-1}+\eta_{I,n-1})(S_{Q,n}+\eta_{Q,n})(S_{I,n'}+\eta_{I,n'})(S_{Q,n'-1}+\eta_{Q,n'-1})\} \\
&= E\{(S_{I,n-1}S_{Q,n}+S_{I,n-1}\eta_{Q,n}+S_{Q,n}\eta_{I,n-1}+\eta_{I,n-1}\eta_{Q,n}) \\
&\quad (S_{I,n'}S_{Q,n'-1}+S_{I,n'}\eta_{Q,n'-1}+S_{Q,n'-1}\eta_{I,n'}+\eta_{I,n'}\eta_{Q,n'-1})\} \\
&= E\{S_{I,n-1}S_{Q,n}S_{I,n'}S_{Q,n'-1}+S_{I,n-1}S_{I,n'}\eta_{Q,n}\eta_{Q,n'-1} \\
&+ S_{Q,n}S_{Q,n'-1}\eta_{I,n-1}\eta_{I,n'}+\eta_{I,n-1}\eta_{I,n'}\eta_{Q,n}\eta_{Q,n'-1}\} \\
&= S_{I,n-1}S_{Q,n}S_{I,n'}S_{Q,n'-1}+S_{I,n-1}S_{I,n'}\sigma^2\delta_{n,n'-1} \\
&+ S_{Q,n}S_{Q,n'-1}\sigma^2\delta_{n-1,n'}+\sigma^4\delta_{n-1,n'}\delta_{n,n'-1} \qquad (18)
\end{aligned}$$

Term three is given by

$$\begin{aligned}
E\{Z_{I,n}Z_{Q,n-1}Z_{I,n'-1}Z_{Q,n'}\} &= E\{(S_{I,n}+\eta_{I,n})(S_{Q,n-1}+\eta_{Q,n-1})(S_{I,n'-1}+\eta_{I,n'-1})(S_{Q,n'}+\eta_{Q,n'})\} \\
&= E\{(S_{I,n}S_{Q,n-1}+S_{I,n}\eta_{Q,n-1}+S_{Q,n-1}\eta_{I,n}+\eta_{I,n}\eta_{Q,n-1}) \\
&\quad (S_{I,n'-1}S_{Q,n'}+S_{I,n'-1}\eta_{Q,n'}+S_{Q,n'}\eta_{I,n'-1}+\eta_{I,n'-1}\eta_{Q,n'})\} \\
&= E\{S_{I,n}S_{Q,n-1}S_{I,n'-1}S_{Q,n'}+S_{I,n}S_{I,n'-1}\eta_{Q,n-1}\eta_{Q,n'} \\
&+ S_{Q,n-1}S_{Q,n'}\eta_{I,n}\eta_{I,n'-1}+\eta_{I,n}\eta_{I,n'-1}\eta_{Q,n-1}\eta_{Q,n'}\} \\
&= S_{I,n}S_{Q,n-1}S_{I,n'-1}S_{Q,n'}+S_{I,n}S_{I,n'-1}\sigma^2\delta_{n',n-1} \\
&+ S_{Q,n-1}S_{Q,n'}\sigma^2\delta_{n,n'-1}+\sigma^4\delta_{n,n'-1}\delta_{n-1,n'} \qquad (19)
\end{aligned}$$

Term four is given by

$$\begin{aligned}
E\{Z_{I,n}Z_{Q,n-1}Z_{I,n'}Z_{Q,n'-1}\} &= E\{(S_{I,n}+\eta_{I,n})(S_{Q,n-1}+\eta_{Q,n-1})(S_{I,n'}+\eta_{I,n'})(S_{Q,n'-1}+\eta_{Q,n'-1})\} \\
&= E\{(S_{I,n}S_{Q,n-1}+S_{I,n}\eta_{Q,n-1}+S_{Q,n-1}\eta_{I,n}+\eta_{I,n}\eta_{Q,n-1}) \\
&\quad (S_{I,n'}S_{Q,n'-1}+S_{I,n'}\eta_{Q,n'-1}+S_{Q,n'-1}\eta_{I,n'}+\eta_{I,n'}\eta_{Q,n'-1})\} \\
&= E\{S_{I,n}S_{Q,n-1}S_{I,n'}S_{Q,n'-1}+S_{I,n}S_{I,n'}\eta_{Q,n-1}\eta_{Q,n'-1} \\
&+ S_{Q,n-1}S_{Q,n'-1}\eta_{I,n}\eta_{I,n'}+\eta_{I,n}\eta_{I,n'}\eta_{Q,n-1}\eta_{Q,n'-1}\} \\
&= S_{I,n}S_{Q,n-1}S_{I,n'}S_{Q,n'-1}+S_{I,n}S_{I,n'}\sigma^2\delta_{n',n} \\
&+ S_{Q,n-1}S_{Q,n'-1}\sigma^2\delta_{n,n'}+\sigma^4\delta_{n,n'}\delta_{n-1,n'-1} \qquad (20)
\end{aligned}$$

Each of our terms is composed of four products. The first product is the product of signal terms, and if we combine these with appropriate signs, we find that they sum to the square of the mean value of our error signal.

$$\begin{aligned}
E\{L_f^2\} &= S_{I,n-1}S_{Q,n}S_{I,n'-1}S_{Q,n'} - S_{I,n-1}S_{Q,n}S_{I,n'}S_{Q,n'-1} \\
&- S_{I,n}S_{Q,n-1}S_{I,n'-1}S_{Q,n'} + S_{I,n}S_{Q,n-1}S_{I,n'}S_{Q,n'-1} \qquad (21)
\end{aligned}$$

Hence, we remove these terms to find the variance of our error signal.

$$\mathrm{var}\{L_f\} = E\{L_f^2\} - (E\{L_f\})^2$$

$$
\begin{aligned}
&= \sum_{n=2}^{N}\sum_{n'=2}^{N} S_{I,n-1}S_{I,n'-1}\sigma^2\delta_{n,n'} \\
&+ S_{Q,n}S_{Q,n'}\sigma^2\delta_{n-1,n'-1} + \sigma^4\delta_{n-1,n'-1}\delta_{n,n'} \\
&- S_{I,n-1}S_{I,n'}\sigma^2\delta_{n,n'-1} \\
&- S_{Q,n}S_{Q,n'-1}\sigma^2\delta_{n-1,n'} - \sigma^4\delta_{n-1,n'}\delta_{n,n'-1} \\
&- S_{I,n}S_{I,n'-1}\sigma^2\delta_{n',n-1} \\
&- S_{Q,n-1}S_{Q,n'}\sigma^2\delta_{n,n'-1} - \sigma^4\delta_{n,n'-1}\delta_{n-1,n'} \\
&+ S_{I,n}S_{I,n'}\sigma^2\delta_{n',n} \\
&+ S_{Q,n-1}S_{Q,n'-1}\sigma^2\delta_{n,n'} + \sigma^4\delta_{n,n'}\delta_{n-1,n'-1}
\end{aligned} \tag{22}
$$

We can now take the indicated sum over n' bearing in mind that $\delta_{n,n'-1}\delta_{n-1,n'} = 0$ and $\delta_{n-1,n'}\delta_{n,n'-1} = 0$, because of their non-overlapping support.

$$
\begin{aligned}
\mathrm{var}\{L_f\} &= \sum_{n=2}^{N} S_{I,n-1}^2\sigma^2 + S_{Q,n}^2\sigma^2 + \sigma^4 + S_{I,n}^2\sigma^2 + S_{Q,n-1}^2\sigma^2 + \sigma^4 \\
&- \sum_{n=2}^{N-1} S_{I,n-1}S_{I,n+1}\sigma^2 - \sum_{n=3}^{N} S_{Q,n}S_{Q,n-2}\sigma^2 \\
&- \sum_{n=3}^{N} S_{I,n}S_{I,n-2}\sigma^2 - \sum_{n=2}^{N-1} S_{Q,n-1}S_{Q,n+1}\sigma^2
\end{aligned} \tag{23}
$$

For small frequency errors $\Delta f \approx 0$, we may write

$$
\begin{aligned}
S_{I,n} &= \sqrt{C}\cos\Delta\theta_n \\
S_{Q,n} &= \sqrt{C}\sin\Delta\theta_n \\
S_{I,n}^2 + S_{Q,n}^2 &= C \\
S_{I,n-1}^2 + S_{Q,n-1}^2 &= C
\end{aligned} \tag{24}
$$

This allows us to write

$$
\begin{aligned}
\mathrm{var}\{L_f\} &= (N-1)(2C\sigma^2 + 2\sigma^4) \\
&- C\sigma^2\sum_{n=2}^{N-1}(\cos\Delta\theta_{n-1}\cos\Delta\theta_{n+1} + \sin\Delta\theta_{n-1}\sin\Delta\theta_{n+1}) \\
&- C\sigma^2\sum_{n=3}^{N}(\cos\Delta\theta_n\cos\Delta\theta_{n-2} + \sin\Delta\theta_n\sin\Delta\theta_{n-2}) \\
&= (N-1)(2C\sigma^2 + 2\sigma^4) \\
&- C\sigma^2\sum_{n=2}^{N-1}\cos(\Delta\theta_{n-1} - \Delta\theta_{n+1}) \\
&- C\sigma^2\sum_{n=3}^{N}\cos(\Delta\theta_n - \Delta\theta_{n-2})
\end{aligned} \tag{25}
$$

We assume that $\theta_n = \theta_{n-k} + k2\pi\Delta f_D T_{CO}$, and so

$$\begin{aligned}
\mathrm{var}\{L_f\} &= (N-1)(2C\sigma^2 + 2\sigma^4) \\
&- C\sigma^2 \sum_{n=2}^{N-1} \cos(4\pi\Delta f_D T_{CO}) \\
&- C\sigma^2 \sum_{n=3}^{N} \cos(4\pi\Delta f_D T_{CO}) \\
&= (N-1)(2C\sigma^2 + 2\sigma^4) \\
&- (N-2)2C\sigma^2 \cos(4\pi\Delta f_D T_{CO}) \qquad (26)
\end{aligned}$$

For small frequency errors, we may finally write

$$\mathrm{var}\{L_f\} = 2C\sigma^2 + 2(N-1)\sigma^4 \qquad (27)$$

4 Integration Gain of the Loop Filter

All of our work so far can be summarized as follows.

$$\begin{aligned}
E\{L_f\} &= 2\pi(N-1)CT_{CO}\Delta f \\
\mathrm{var}\{L_f\} &= 2C\sigma^2 + 2(N-1)\sigma^4 \\
\sigma^2 &= \frac{N_0}{2T_{CO}} \qquad (28)
\end{aligned}$$

These results can be used to calculate the variance of the normalized error signal.

$$\begin{aligned}
\mathrm{var}\{\Delta f\} &= \frac{\mathrm{var}\{L_f\}}{\left(\frac{\partial E\{L_f\}}{\partial \Delta f}\right)^2} \\
&= \frac{2C\sigma^2 + 2(N-1)\sigma^4}{(2\pi)^2(N-1)^2C^2T_{CO}^2} \\
&= \frac{2C\sigma^2\left(1 + \frac{2(N-1)\sigma^4}{2C\sigma^2}\right)}{(2\pi)^2(N-1)^2C^2T_{CO}^2} \\
&= \frac{1}{(2\pi)^2(N-1)^2\frac{C}{N_0}T_{CO}^3}\left(1 + \frac{N-1}{\frac{2CT_{CO}}{N_0}}\right) \qquad (29)
\end{aligned}$$

When we close a discrete time loop around such measurements, the integration gain of the filter reduces the variance of the frequency errors.

$$\begin{aligned}
\mathrm{var}\{\Delta f\} &= \frac{2\mathrm{var}\{L_f\}B_{f,1}NT_{CO}}{\left(\frac{\partial E\{L_f\}}{\partial \Delta f}\right)^2} \\
&= \frac{B_{f,1}N}{2\pi^2\frac{C}{N_0}(N-1)^2T_{CO}^2}\left(1 + \frac{N-1}{\frac{2C}{N_0}T_{CO}}\right) \qquad (30)
\end{aligned}$$

In this equation, NT_{CO} is the sampling time of the loop, and B_1 is the one sided bandwidth of the loop.

5 Discussion

These results are best understood in the following equivalent forms.

$$
\begin{aligned}
\mathrm{var}\{\Delta f\} &= \frac{B_{f,1}N}{2\pi^2(N-1)^2T_{CO}^2C/N_0}\left(1+\frac{N-1}{2(C/N_0)T_{CO}}\right) \\
\mathrm{var}\{\Delta f\} &= \frac{B_{f,1}N^3}{2\pi^2(N-1)^2T_B^2C/N_0}\left(1+\frac{N(N-1)}{2(C/N_0)T_B}\right)
\end{aligned}
\tag{31}
$$

The term inside the brackets is called squaring loss. At high signal-to-noise ratios, where GPS receivers typically operate, the second term inside the brackets is less than one. In other words, squaring loss is not important. At low signal-to-noise ratios, this second term dominates and squaring loss becomes important.

The first equation continues to use N as the number of non-coherent samples, and T_{CO} as the coherent integration time. For a fixed loop bandwidth, $B_{f,1}$, the high SNR variance is inversely proportional to C/N_0, N and T_{CO}^2. The low SNR variance is inversely proportional to $(C/N_0)^2$ and T_{CO}^3. It is more or less independent of N.

The second equation above recognizes that $NT_{CO} = T_B$, where T_B is the duration of a data bit. We use this relationship to remove the dependence on T_{CO}. After all, if we specify N and T_B, then T_{CO} is known. In this case, the high SNR variance is proportional to N and inversely proportional to C/N_0 and T_B^2. The low SNR variance is proportional to N^3, and inversely proportional to $(C/N_0)^2$ and T_B^3.

In all cases, coherent averaging is more effective than non-coherent averaging. Sometimes, this is dramatically true. Clearly, the coherent averaging time should be as large as possible. When the signal has been acquired and the locations of the data bit boundaries are well known, then large values of T_{CO} can be used. Typically, $T_{CO} = 0.5T_B$ and $N = 2$. The bit boundary times are not known during signal acquisition, and large values of T_{CO} are risky. After all, if a bit transition occurs in the middle of the coherent average, then the average will probably be worthless. Hence, smaller values of T_{CO} are used and larger values of N result. Typically, $T_{CO} = 0.1T_B$ and $N = 10$. For GPS, this means that the coherent average is only 2 ms during signal code acquisition, but extends to 10 ms during tracking.