Chapter 10

Signal-to-Noise Ratio and Ranging Precision

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Absent obstructions or reflections, today’s GPS receiver can measure pseudoranges to a satellite with a precision of 0.5 meters or better. In the last chapter, we attributed this ranging prowess to the auto-correlation functions of the C/A-codes and more specifically to the slope of the main peak of that function. In this chapter, we will substantiate that claim with a quantitative analysis of ranging precision in the presence of white noise. We will discover that the ranging precision does indeed depend on the slope of the correlation function, which in turn depends on the bandwidth of the signal.

As we shall see, the ranging performance also depends on the signal-to-noise ratio, $C/N_0$, and the averaging time used by the receiver. $C/N_0$ is the ratio of the power in the received signal to the power spectral density of the competing noise. Signal power, $C$, has units of watts or joules/sec. Power spectral density, $N_0$, has units of watts/hertz. Hence, $C/N_0$ has units of hertz.
The power available for the C/A-code on the satellite is approximately 27 watts, but the power collected by a typical receiver on the earth’s surface is only $10^{-16}$ watts or so. When received, the GPS signal is swamped by the noise in the front end of the receiver. In this chapter, we develop estimates of the received signal power and the received noise power spectral density. The ratio of these two powers is key to ranging precision.

We proceed as follows. Sections 10.1 and 10.2 begin by analyzing the power in the GPS signal. Section 10.1 shows that the majority of the signal attenuation is due simply to loss introduced by the long path from the satellites to the surface of the earth. In this section, we also show how the transmitting antennas on the satellites are used to ameliorate some of this loss. Section 10.2 discusses the role of the receiver’s antenna and estimates the received signal power as a function of satellite elevation relative to the receiver.

Sections 10.3 and 10.4 turn our attention to the power in the competing noise. Section 10.3 introduces the language used to discuss noise and shows how to compute the aggregate noise for a system with multiple subsystems—this latter result is called Friis’ formula. The work in Section 10.3 is applicable to any radio system, and Section 10.4 applies these general results to GPS.
Sections 10.5 and 10.6 analyze the performance of GPS ranging when additive white noise is the only disturbance. They teach the advantages of spread spectrum signaling when the goal is high precision ranging. To do so, they introduce the delay lock loop (DLL), which correlates the incoming signal with replicas of the signal generated by the receiver. As described in Section 9.8.1, this process is called de-spreading. As the name suggests, de-spreading concentrates the GPS signal power in a narrow bandwidth, where the GPS signal is stronger than the noise. Section 10.6 contains much of the detailed analysis and can be omitted on a first reading. Section 10.7 shows how spread spectrum signals also enable precise ranging in the presence of reflected signals called multipath. Finally, Section 10.8 is a brief summary.

### 10.1 Signal Path Loss and Transmit Antenna Gain

In the year 2005, a GPS satellite dedicates approximately 27 watts of power for the C/A-code signal on L1 [Aparicio et al. (1996)]. In decibels, this value is equal to $10\log_{10} 27 = 14.3$ dBW and appears at the top of Table 10.1, which summarizes the analysis of this section.

The transmitted signal power is limited by cost. Ultimately, the signal power that serves our earthbound population of users comes from the solar panels that are carried by the satellite. When the satellite is in eclipse, the signal power comes from onboard batteries that were charged when the satellite was in sunlight. Increased signal power demands larger solar arrays and batteries, and the cost to launch is a fast function of the satellite weight. GPS signal power is also limited by the need to coexist with other GNSS systems operating in the same band. Recall that GPS uses code division multiple access to identify satellites, but this code division capability can be overwhelmed if one satellite is much stronger than another.

If this C/A signal power were broadcast in all directions, then the power spatial density at radius $R$ meters would be $1/4\pi R^2$ times the radiated power. This term is the so-called path loss or spreading loss. As shown in Figure 10.1, this term accounts for the spreading of the total energy over the surface area of the sphere centered on the satellite. If the satellite antenna truly broadcast its energy uniformly in all directions, then the received power density at the surface of the earth would be

![Figure 10.1 Power spatial density of the GPS signal. A power spatial density has units of watts per unit area.](image-url)
\[
\mathcal{P} = \frac{P_T}{4\pi R^2} \text{ watts/m}^2
\]
\[
\mathcal{P}_{db} = P_{T,db} - 10 \log_{10} R \text{ dBW/m}^2
\]
\[
P_T = 27 \text{ watts}
\]
\[
R = \text{distance from satellite to user in meters}
\]

Do not confuse \(P_T\) and \(\mathcal{P}\). \(P_T\) is a power measured in watts, while \(\mathcal{P}\) is a power spatial density measured in watts/m\(^2\). \(P_{T,db}\) and \(\mathcal{P}_{db}\) are the same quantities measured in dBW and dBW/m\(^2\), respectively.

The altitude of the satellite is approximately 20,190 kilometers, and so spreading loss will be appreciable. As shown in Figure 10.2, distance from the satellite to a user depends on the user location on the earth. In fact, the satellite to user range can be calculated as a function of the satellite’s elevation angle, \(el\), at the user.

\[
R = -R_E \sin el + \sqrt{R_E^2 \left(\sin^2 el - 1\right) + R_{SV}^2}
\]

\[
R_E = 6371 \times 10^3 \text{ m}
\]

\[
R_{SV} = 26,560 \times 10^3 \text{ m}
\]

\(el = \text{satellite elevation angle at the user}\)

Even when the satellite is directly over the head of the user, the range is 20,190 km and the corresponding path loss is \(-157.1\) dB! When the satellite is only 5° above the user’s horizon, this range increases to 25,240 km and the path loss increases to \(-159\) dB. These losses correspond to a power attenuation of approximately \(1.6 \times 10^{-16}\). No wonder the received signal is so weak!

Some of this power can be recovered because the satellite can focus its energy towards the earth. This benefit is quantified relative to the power density from an antenna that radiates uniformly in all directions. Antenna gain gives amplification of the power density in a given direction relative to that predicted by \(1/4\pi R^2\) for an omni-directional (or isotropic) antenna.

In general, this gain depends on the size and design of the satellite antenna, but we do not
consider those complexities here. Rather, we estimate the satellite antenna gain by considering Figure 10.3, which shows a sphere centered on a GPS satellite. We assume that the satellite is able to concentrate its radiated power within the solid angle $2\alpha$ measured away from the vector that connects the satellite to the center of the earth. The angle $\alpha$ is called the nadir angle. As shown in Figure 10.2, the nadir angle can be related to the elevation angle, $el$, of the satellite relative to the user’s horizon.

With such focusing, the concentration factor (transmitter antenna gain) is given by the ratio of the area of the sphere to the area of the spherical cap created by the angle $\alpha$. That ratio is given by

$$G_T(\alpha) = \frac{4\pi R^2}{\pi p^2} = \frac{2}{1 - \cos \alpha}$$

(10.3)

The details are given in Figure 10.3. The earth subtends an angle $\pm 13.9^\circ$ as seen from a GPS satellite, but the GPS beam has a somewhat wider spread of $\pm 21.3^\circ$ for the L1 signal. Consequently, the gain may be approximated as $10\log_{10} G_T(21.3^\circ) = 14.7$ dB.

The actual antenna gain for the GPS transmit antenna is smaller than this approximation for two reasons. First, there is additional loss in the antenna itself that suppresses the radiated power. Second, the gain is tailored to compensate for the greater distance to the users at the edge of the earth (as seen by the satellite). The satellite antenna gain is approximately 2 dB stronger at the edge of coverage ($\alpha = 13.9^\circ$) than along the so-called bore sight ($\alpha = 0^\circ$).

For a satellite at low elevation, the antenna gain is around 12.1 dB, and the effective radiated power in the direction of such a user is 437 watts. For a satellite at 40° elevation, the gain is approximately 12.9 dB and the effective radiated power in this direction is effectively 525 watts. Finally, for a satellite at zenith, the gain is approximately equal to 10.2 dB and the effective radiated power is 282 watts.
When the factors described above are combined, the received power density is given by

\[ P = \frac{P_T G_T}{4\pi R^2 L_A} \text{ watts/m}^2 \]

\[ P_{\text{dB}} = P_{T,\text{dB}} + G_{T,\text{dB}} - 20 \log_{10} R - 11 - L_{A,\text{dB}} \text{ dBW/m}^2 \]  

(10.4)

Equation (10.4) includes a term, \( L_A \), to model power loss as the signal propagates through the atmosphere. As shown, we use a value of 0.5 dB in Table 10.1 for this loss.

Our work so far is summarized in Table 10.1 and Figure 10.4. Radio engineers refer to such calculations as link budgets.

### 10.2 Received Signal Power and Receiver Antenna Gain

The power in the incident signal field is captured by the receiver’s antenna. In fact, the received power is equal to the power density of the incident field times the effective area of the receive antenna. The effective area, \( A_E \), measures the antenna’s ability to capture the power in a field incident from a certain direction, and the gain, \( G_R \), measures the antenna’s ability to focus transmitted power in a certain direction. Remarkably, the effective area of any antenna is related to its gain. An antenna’s ability to capture power from a certain direction is proportional to its ability to send power in that same direction. This reciprocity is skillfully explored in the classic textbook by Jordan and Balmain (1968), and is given by

\[ A_E = G_R \lambda^2 / 4\pi \]  

(10.5)

In this equation, \( \lambda \) is the wavelength of the signal and is given by \( \lambda = c/f \), where \( c \) is the speed of light and \( f \) is the frequency of the signal.
An isotropic antenna is equally sensitive to signals coming from any direction. Such an antenna has unit gain, $G_R = 1$, for all azimuth and elevation angles and $A_E = \lambda^2 / 4\pi$. For such an antenna, the received signal power is given by

$$P_R = \frac{P_T G_T}{L_A} \left( \frac{\lambda}{4\pi R} \right)^2 \text{watts} \quad (10.6)$$

For GPS, these powers are shown in Table 10.2, which shows that the received power is a mild function of the satellite’s elevation angle. This relationship is also plotted in Figure 10.5 for all elevation angles from $5^\circ$ to zenith.

In fact, the antennas used by GPS receivers are not isotropic because they only need to receive signals from above the user’s horizon. Above the horizon, they must provide nearly full sky coverage, because GPS satellites are sprinkled across the heavens, and good performance requires that satellites in all directions be received. Otherwise, the geometric dilution of precision will grow and positioning accuracy will deteriorate (see Section 6.1.2). Like isotropic antennas, the gain does not vary with azimuth because GPS satellites are found at all points of the compass. Unlike isotropic antennas, the gain does vary with elevation angle, and this variation is captured in the elevation pattern.

Such an elevation pattern is shown in Figure 10.6 for a patch antenna that measures approximately $10 \text{ cm} \times 10 \text{ cm} \times 1 \text{ cm}$, and is relatively low cost. Patch antennas can be smaller, and so they are used in a wide variety of applications. As shown, gain, $G_R$, decreases slowly

<table>
<thead>
<tr>
<th>Table 10.2 Received signal power for C/A-code signal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Satellite at Low Elevation</strong> ($el = 5^\circ, \alpha = \pm 13.9^\circ$)</td>
</tr>
<tr>
<td>Received power density ($PD_S$)</td>
</tr>
<tr>
<td>= $-133.1 \text{dBW/m}^2$</td>
</tr>
<tr>
<td>Effective area of an omni-directional receive antenna ($\lambda^2 / 4\pi$)</td>
</tr>
<tr>
<td>Receive power available from an isotropic antenna</td>
</tr>
<tr>
<td>Gain of a typical patch receive antenna ($G_R$) relative to isotropic antenna</td>
</tr>
<tr>
<td>C/A-code received power available to a typical receive antenna</td>
</tr>
</tbody>
</table>
from approximately +4 dBic at zenith to –4 dBic at an elevation angle of 5°. The notation dBic means gain relative to an isotropic antenna. Gains from Figure 10.6 are used to modify the receiver signal powers shown in Table 10.2.

GPS engineers use the symbol $C$ to denote the received signal power inclusive of the gain from the receiver antenna, $G_R$, and any losses from the receiver, $L_R$. With these added factors, we have

$$C = \frac{P_R G_R}{L_R}$$  \hspace{1cm} (10.7)

If we substitute for $P_R$, then

$$C = \frac{P_T G_T G_R}{L_A L_R} \left( \frac{\lambda}{4\pi R} \right)^2 = \frac{P_T G_T G_R}{L_A L_R} \left( \frac{c}{4\pi R f} \right)^2 \text{watts} \hspace{1cm} (10.8)$$

Table 10.2 gives numerical values for this power in the GPS civil signal at the L1 frequency.

The frequency dependence in (10.8) was a critical consideration when choosing the GPS carrier frequencies. The receiving antenna must be small and able to capture signals coming from all directions. Hence, it cannot greatly concentrate or amplify the signal coming from any given direction. Since $G_R \approx 1$, the frequency cannot be too high. Otherwise, the received signal would be too weak. Satellite systems that only need to track one satellite have greater flexibility when choosing their carrier frequencies. Frequently, these systems can employ receiving antennas with higher directionality and gain, and can therefore afford higher frequencies.

Of course, a host of other factors affected the frequency selection for GPS. The GPS signal needed to fall within the available radio spectrum that could be used for space-to-earth signaling. In addition, the frequency could not be too low—otherwise ionospheric delays become too unpredictable.

Most civilian GPS receive antennas generate patterns similar to the one shown in Figure 10.6. However, the exceptions are both important and interesting. For example, some antennas
place beams on the individual GPS satellites, and thus strengthen the GPS signals. Some adapt their patterns to attenuate undesired signals that may interfere with GPS. Others discriminate against signals that have polarizations other than the one used by GPS. Needless to say, these adaptive antennas are significantly more sophisticated than the simple antenna characterized by Figure 10.6. Even so, they are used in military applications for protection against intentional interference. We have more to say about these adaptive antennas in Chapter 13.

Other antennas are designed to reject multipath at fixed sites like differential GPS reference stations, where multipath is particularly worrisome [Counselman (1999) and dBSys-
tems (2000)]. These multipath-limiting antennas (MLAs) are taller than the patch antenna described above. Their height precludes their use on most mobile platforms, but also enables a much more rapid roll off of gain with elevation angle. This rapid roll off allows the antenna to attenuate signals that are reflected from below the ground. This is further discussed in Section 10.7.3.

As shown in Table 10.2, the received GPS signals are certainly weak, but the competing signals are also usually weak. The competitors include:

- thermal noise generated in the receiver
- natural noise from sources outside of the receiver
- reflected signals
- signals from other GPS satellites
- man-made signals from systems other than GPS.

Of these, the first two are considered in the next four sections. More specifically, Section 10.3 provides a general introduction to noise. Section 10.4 applies the findings from Section 10.3 to GPS. Sections 10.5 and 10.6 introduce the delay lock loop and analyze its performance in the presence of noise. Section 10.7 moves on to describe the impact of signal reflections on
the same delay lock loop. As described in Chapter 9, the multiple-access interference between GPS satellites is minimized by careful selection of the spread spectrum sequences. More thorough analyses of this inter-satellite interference are contained in Sarwate and Pursley (1980). Interference from other man-made radio sources is the focus of Chapter 13.

### 10.3 Noise

Noise competes with a man-made signal at the receiver in almost all radio systems. In this chapter, we concern ourselves with *white noise*. Recall from Chapter 8 that white noise combines noise components from all frequencies with equal strength, and so it has a power spectral density that is a constant $N_0/2$ W/Hz. White noise is an excellent model for the natural noise that is received along with GPS signals, because natural noise has a constant power spectral density across the GPS band. However, white noise may not be a good model for man-made signals that may find their way into the GPS band. We return to this subject in Chapter 13.

Our power spectral density, $N_0/2$, should not be confused with the power spatial density, $P$, used in the last two sections. Power spectral density measures the power per unit of bandwidth—it has units of watts per hertz. Power spatial density measures the amount of power in a unit of area—it has units of watts per meter$^2$.

The power spectral density is denoted $N_0/2$ rather than $N_0$ for convenience. Consider Figure 10.7. The top half shows $|H(f)|^2$ for a low-pass filter, where $H(f)$ is the transfer function for the filter. The bottom half shows $|H(f)|^2$ for a bandpass filter. White noise is input to both

![Figure 10.7](image_url)

**Figure 10.7** Noise power spectral density. Power spectral density (PSD) has units of watts per hertz. The notation $N_0/2$ preserves the relationship $P_N = N_0B_n$ for low-pass filters and bandpass filters. The parameter $n$ refers to the order of a Butterworth filter, which is described in Section 8.5.8.
filters, and we wish to calculate the noise power at the output, \( P_N \).

We usually characterize low-pass filters with one-sided bandwidths because it is sensible to measure the frequency response from zero frequency up to some frequency where the filter response is weak. Since we are concerned with noise power, it also makes sense to consider the noise equivalent bandwidth described in Chapter 8. If we use the one-sided noise equivalent bandwidth, \( B_{1,N} \), then the power in the output noise is given by the following simple relationship.

\[
P_N = 2 B_{1,N} \frac{N_0}{2} \text{watts} = B_{1,N} N_0
\]

(10.9)

For the bandpass filter, we tend to rely on two-sided bandwidths because it is sensible to span the frequencies with a strong response. The two-sided noise equivalent bandwidth, \( B_{2,N} \), is shown in Figure 10.7. The power in the output noise is given by

\[
P_N = 2 B_{2,N} \frac{N_0}{2} \text{watts} = B_{2,N} N_0
\]

In both cases, we end up with the simple relationship, \( P_N = B_N N_0 \).

In this chapter and all of the subsequent relationship, \( P_N = B_N N_0 \).

In this chapter and all of the subsequent ones, we will discover that the ratio of the signal power, \( C \), to the total noise power spectral density is a key system parameter. We will denote this ratio as \( C/N_0 \). As described in Section 10.3.1, noise figure computes the degradation of \( C/N_0 \) as the signal passes through receiver components that add their own noise to the system. In the chapters that follow, we shall also discover that the bandwidth of a GPS receiver is wider for the sections nearest the antenna and becomes narrower as the processing develops. For any given noise-equivalent bandwidth, \( B_N \), the signal power to noise power ratio, \( C/P_N \), is

\[
\frac{C}{P_N} = \frac{C}{B_N N_0}
\]

(10.10)

### 10.3.1 Noise Temperature and Noise Figure

In a given bandwidth, \( B_N \), noise power is also related to an equivalent noise temperature, \( T_{eq} \), as follows:

\[
P_N = k T_{eq} B_N \text{ watts}
\]

\[
k = 1.38 \times 10^{-23} \text{ J/K = Boltzmann’s constant}
\]

\[
T_{eq} = \text{equivalent temperature}
\]

\[
N_0 = k T_{eq} \text{ watts/Hz}
\]

(10.11)

At first, the use of a temperature to describe noise power spectral density may not seem intuitive. However, this use finds its sensibility when considering thermal noise, where the equivalent temperature is simply the physical temperature of the device generating the noise. At any temperature above absolute zero (0 K), thermal noise is created by the inevitable motion of charge carriers within any conductor or semiconductor. The resulting noise power is proportional to the physical temperature of the device.
However, Equation (10.11) is also used for noise sources that have little dependence on the actual physical temperature—they are not thermal noise sources at all. For example, antenna temperature is not the physical temperature of the antenna, but simply the temperature of a thermal noise source that would provide the same noise power. For many radio systems, the antenna temperature models noise from the sky and from the warm earth. For GPS, the antenna temperature tends to be below the ambient temperature. For radio systems that operate at lower frequencies, the antenna temperature can be much higher than the ambient temperature. From this point forward, we will drop the adjective ‘equivalent’ when describing noise temperatures and understand that the noise source may or may not be thermal.

For many systems, including GPS, the total system noise is a blend of noise from the antenna and noise from subsystems within the receiver. A generic subsystem is shown in Figures 10.8 and 10.9. The noise at the subsystem output is the sum of internal noise and amplified input noise, where the internal noise may be due to any noise producing mechanism within the subsystem. To simplify analysis, this internal noise can be described as a second noise temperature at the input. This second noise temperature is said to be referenced to the input and is denoted $T_E$ for effective input temperature. Then the total noise power density at the output is simply $Gk(T_{in} + T_E)$, where $G$ is the power gain of the subsystem.

As mentioned above, noise figure, $F$, computes the degradation of $C/N_0$ as the signal passes through receiver components that add their own noise to the system.

$$F(T_{in}) = \frac{(C/N_0)_{in}}{(C/N_0)_{out}}$$

$$= \frac{CGk(T_{in} + T_E)}{GkT_{in}}$$

$$= 1 + \frac{T_E}{T_{in}} \geq 1$$

$$T_E = (F(T_{in}) - 1)T_{in} \quad (10.12)$$
$T_E$ is the effective temperature of the internal noise source. If there is no internal noise, then $F = 1$ and the signal-to-noise ratio at the output of the subsystem is equal to the signal-to-noise ratio at the input. If there is internal noise, then the noise figure is greater than unity and the signal-to-noise ratio is degraded at the output.

There is no one-to-one relationship between noise figure and effective temperature unless the input temperature is known or specified. When writing a noise figure specification for a radio device or subsystem, the manufacturer does not know $T_{in}$ because it depends on the application. Consequently, noise figure specification for components are often based on the assumption that the input noise temperature is room temperature or $T_0 = 290$ K. This protocol yields the following one-to-one relationship between $F(T_0)$ and $T_E$.

$$F(T_0) = 1 + \frac{T_E}{290} \quad T_E = (F(T_0) - 1)290 \text{ K}$$

This relationship allows radio components to be characterized either by their noise figure or effective temperature. However, it must be used with care because

$$\frac{(C/N_0)_{in}}{(C/N_0)_{out}} = F(T_{in}) \neq F(T_0)$$

### 10.3.2 Noise in a Cascade of Subsystems

Any radio receiver is a cascade of subsystems or components. For noise analysis, all of these components are characterized by their power gain and noise figure. As we shall discover, the components found nearest to the receiving antenna determine the noise performance of the receiver.

Some of the subsystems amplify power. Some attenuate the signal and are called **passive**. The passive subsystems include cables and connectors between subsystems. The gain for these
elements is less than one, \( G < 1 \). Passive elements dissipate the lost power as heat and therefore introduce thermal noise. Conveniently, the noise figure for a passive element is equal to the power loss, \( L \), which is the inverse of gain [Tsui (1995) and Vizmuller (1995)].

\[
F = L = \frac{1}{G} > 1
\]

\[
T_E = \left( \frac{1 - G}{G} \right) \times 290
\]  

(10.15)

Homework Problem 10-2 asks you to prove this relationship. Passive elements inevitably degrade the signal-to-noise ratio. The receiver design goal is to keep the loss small—especially for the components near the antenna. After all, if the gain is 0.9, then the effective temperature of the device is only \( 290/9 \approx 32 \) K. However, if the gain is 0.1, then the effective temperature rises to \( 9 \times 290 \approx 2610 \) K.

Our task in this section is to find the aggregate noise figure of a receiving system from the specifications for the individual components. Consider Figure 10.10, which shows a cascade of subsystems. The input contains signal power, \( C \), and noise power spectral density, \( kT_A \), where \( T_A \) is the effective noise temperature of the external noise received by the antenna. Each subsystem is characterized by: a gain or loss, \( G_i \) or \( L_i \); and a noise figure or effective temperature, \( F_i \) or \( T_{E,i} \). If the element is passive, then \( F_i = L_i \).

As shown in Figure 10.10, the signal power at the end of the chain is \( G_1 G_2 G_3 C \), and the noise density at the end of the cascade is

\[
N_{0,3} = kG_1 G_2 G_3 (T_A + T_R)
\]

\[
T_R = T_{E,1} + \frac{T_{E,2}}{G_1} + \frac{T_{E,3}}{G_1 G_2}
\]  

(10.16)

This result is called Friis’ formula. \( T_R \) is the effective temperature of all of the noise sources that are internal to the receiver. \( T_A + T_R \) sums the external noise with the internal noise.

The noise figure of the entire receiver may also be written using the noise figures for the individual subsystems as follows.

\[
F_R = F_1 + \frac{(F_2 - 1)}{G_1} + \frac{(F_3 - 1)}{G_1 G_2}
\]  

(10.17)

With these results, \( C/N_0 \) may be written in a way that includes the effect of the external noise and the noise that is internal to the radio system.

\[
\left( \frac{C}{N_0} \right)_\text{out} = \frac{C}{k \left( T_A + T_{E,1} + \frac{T_{E,2}}{G_1} + \frac{T_{E,3}}{G_1 G_2} \right)}
\]

\[
= \frac{C}{k(T_A + T_R)}
\]  

(10.18)

These formulas reveal much that is of interest. The antenna temperature and the tempera-
ture of the first subsystem contribute directly to the overall effective temperature. However, the noise contributions of sections later in the cascade may or may not be significant depending on the gains of the preceding subsystems. If $G_1$ is large, then the relative importance of the internal noise from subsystems 2 and 3 will be diminished. If $G_2$ is the first large gain, then the noise from subsystems 1 and 2 will be important, but the noise from section 3 will be attenuated. In most radio receivers, gain is distributed throughout the cascade. Hence, the noise performance of the entire radio is dominated by the noise performance of the first few sections encountered by the received signal. This makes sense because noise in these early sections will be amplified by all of the following sections.

We can also write $\frac{C}{N_0}$ as a function of the noise figure of the overall receiving chain, $F_R$:

$$\left( \frac{C}{N_0} \right)_{\text{out}} = \frac{1}{F_R(T_A)} \left( \frac{C}{N_0} \right)_{\text{in}}$$

$$= \frac{C}{kT_A F_R(T_A)}$$

(10.19)

If we include (10.7) and use decibels, we may write

$$\left( \frac{C}{N_0} \right)_{\text{out}} = P_{R, \text{dB}} + G_{R, \text{dB}} - L_{R, \text{dB}} - 10 \log_{10}(kT_A) - F_{R, \text{dB}}(T_A) \quad \text{dB-Hz}$$

(10.20)

If we use a noise figure that is referenced to an input temperature other than the temperature of our antenna, then we must write

$$\left( \frac{C}{N_0} \right)_{\text{out}} = \frac{C}{kT_0 F_R(T_0) + k(T_A - T_0)}$$

(10.21)

Note the correction term, $k(T_A - T_0)$, in the last equation if $T_A \neq T_0$. 

Figure 10.10 Noise analysis for a cascade of subsystems.
10.4 Noise Analysis of a GPS Receiver

We now apply the general results from Section 10.3 to GPS. Table 10.3 and Figure 10.11 will guide our analysis. The figure shows external noise entering the receiver through the antenna and the first few stages of a generic GPS receiver. The noise generated in these early stages blends with the noise from the antenna to create the overall noise floor for this receiver. The antenna temperature is approximately 75–100 K due to noise received from the sky plus ground radiation. Table 10.3 summarizes the noise contribution from the components in the receiver front end, and is typical for a GPS receiver.

The first and third receiver elements listed in the table are passive components. Their noise figures will be equal to their losses and their effective temperatures are given by (10.15).

Table 10.3 Typical noise characterization of the components in the front end of a GPS receiver

<table>
<thead>
<tr>
<th>Component</th>
<th>Cable and filter that precede LNA</th>
<th>Low-noise amplifier (LNA)</th>
<th>Cable that follows LNA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain G</td>
<td>0.8 = −1 dB</td>
<td>100 = 20 dB</td>
<td>0.1 = −10 dB</td>
</tr>
<tr>
<td>Loss L = 1/G</td>
<td>1.26 = 1 dB</td>
<td>0.01 = −20 dB</td>
<td>10 = 10 dB</td>
</tr>
<tr>
<td>Noise figure F</td>
<td>1.26 = 1 dB (equal to loss)</td>
<td>2 = 3 dB (from manufacturer’s specification)</td>
<td>10 = 10 dB (equal to loss)</td>
</tr>
<tr>
<td>Effective temperature</td>
<td>$T_E = (F - 1)290$ K</td>
<td>$290$ K</td>
<td>$2610$ K</td>
</tr>
</tbody>
</table>

Figure 10.11 Noise analysis for the front end of a GPS receiver. For GPS, the noise is a blend of external noise and noise generated inside the receiver.
The first component is a filter to remove signals outside the GPS band with a cable that connects the antenna to the low-noise amplifier (LNA). This filter is simple and designed for low loss. Moreover, the cable also has low loss because the LNA is located next to the antenna and so the cable is short. For these reasons, we model the loss as 1 dB. The noise figure and temperature are correspondingly small.

To mitigate the noise temperature of the second cable, the LNA is designed for high gain and low noise figure. The table gives a typical value for a GPS LNA. As shown, we assume that the noise figure is 3 dB and the gain is 100 or 20 dB.

The effective temperature of the entire GPS cascade, including the antenna temperature is

\[
T_A + T_R(F_2, G_1) = T_A + \left( \frac{1}{G_1} - 1 \right) 290 + \frac{(F_2 - 1)290}{G_1} + \frac{\left( \frac{1}{G_3} - 1 \right)290}{G_1 G_2}
\]

\[
= T_A + 290 \left( \frac{F_2}{G_1} - 1 \right)
\]

\[
N_0 = 10 \log_{10} k(T_A + T_R(F_2, G_1)) \text{ dBW/Hz}
\]

\[
N_0(F_2 = 3 \text{ dB}, G_1 = 0.8, T_A = 100 \text{ K}) \approx -201.3 \text{ dBW/Hz}
\]

\[
N_0(F_2 = 4 \text{ dB}, G_1 = 0.8, T_A = 100 \text{ K}) \approx -200.0 \text{ dBW/Hz}
\]

As shown, the receiver noise floor depends on three critical parameters—the antenna tem-

![Figure 10.12](image)

**Figure 10.12** Measured C/N₀ versus elevation angle (courtesy of Frank Bauregger, Novariant, Inc.).
temperature (external noise), the gain of the cable and filter that precede the LNA, and the noise figure of the LNA. Thankfully, the loss in the second cable is not very important provided the LNA has reasonably high gain.

We now turn our attention to $C/N_0$ for GPS. Recall from Table 10.2, that the effective carrier power, $C$, in the civil signal varies from $-162.5$ dBW to $-154.5$ dBW for satellites at low elevation and zenith, respectively. As discussed above, the white noise power density is approximately $-201$ dBW/Hz, and so $C/N_0$ ranges from $38.5$ up to $46.5$ dB-Hz. This situation is shown in Figure 10.12, which plots measured $C/N_0$ versus zenith angle. The plot includes the impact of the receive antenna gain to show how this gain influences $C/N_0$. The signal power is approximately 7000 to 45,000 times stronger than the noise power in a 1-Hz bandwidth.

As we mentioned earlier, the bandwidth of a GPS receiver is wider for the sections nearest the antenna and becomes narrower as the processing develops. In fact, the earliest filters in the receiver front end have bandwidths of tens of megahertz. In a 20-MHz bandwidth, the noise power, $P_N = N_0 \times 20 \times 10^6$, is some 450 to 7000 times stronger than the signal power. In decibels, the GPS power is 26.5 to 34.5 dB weaker than the noise power. Radio engineers say that the GPS signal is below the noise floor.

As the signal travels deeper into the receiver, it eventually reaches the delay lock loops described in the next section. The delay lock loops contain correlators that dramatically de-spread the signal bandwidth. After de-spread, the GPS signal power is packed into a null-to-null bandwidth of 100 hertz. In this bandwidth, the GPS signal is approximately 21.5 dB above the noise floor. Table 10.4 lists signal-to-noise ratios as a function of bandwidth.

Figure 10.13 also depicts the effect of de-spread the GPS signal. Both traces show the GPS noise floor of $-201$ dBW/Hz. However, the units have been converted to dBW/MHz.

<table>
<thead>
<tr>
<th>Table 10.4 Signal-to-noise ratios as a function of bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^\circ$ elevation</td>
</tr>
<tr>
<td>----------------------</td>
</tr>
<tr>
<td>Received power in C/A-code signal (PC)</td>
</tr>
<tr>
<td>Noise power density ($N_0$) for a 3 dB LNA noise figure</td>
</tr>
<tr>
<td>$C/N_0$</td>
</tr>
<tr>
<td>$C/P_N = \frac{C}{N_0BW} = \frac{C}{N_020 \times 10^6}$ (20 MHz bandwidth)</td>
</tr>
<tr>
<td>$C/P_N = \frac{C}{N_0BW} = \frac{C}{N_02 \times 10^6}$ (2 MHz bandwidth)</td>
</tr>
</tbody>
</table>
Thus the noise floor has increased by 60 dB to a level of –141 dBW/MHz, because the noise power in 1 MHz is $10^6$ times the power in 1 Hz. The top and bottom traces use results from Section 9.6 to show the power spectral density of the GPS signal before and after de-spreading. Before de-spreading, the GPS signal is much weaker than the noise. After de-spreading, it is 18.5 to 26.5 dB above the noise floor.

10.5 Delay Lock Loops and Ranging Precision

A GPS receiver tracks the distinct and sharp peak of the code auto-correlation function. The resulting pseudorange estimate is unambiguous because the main peak is noticeably larger than any side peaks. The pseudorange estimate is precise because the peak is narrow and the correlation measurements are very sensitive to the location of this sharp event. Indeed, the correlation peak is a triangle with a base width of 600 meters. As mentioned in Chapter 9, the receiver can resolve the arrival time to approximately 0.1% of this base width or 0.6 meters.

In this section, we quantify the performance of spread spectrum codes for ranging. To simplify our analysis, we think of the signal as consisting of code alone. We dispense with the complications of the carrier and data message. We also introduce the basic delay lock loop (or DLL). After signal acquisition, most GPS receivers use a DLL to track the signal from each GPS satellite [Spilker (1996), Van Dierendonck (1996), Holmes (1990)]. This fascinating structure is shown in Figures 10.14 and 10.15. The two figures are similar. Let’s begin with Figure 10.14 and transition our attention to Figure 10.15 as the analysis matures.

As shown, the delay lock loop correlates the received signal with a slightly early replica.
of the signal and a slightly late replica of the signal. When locked to the received signal, the early correlator samples the peak of the correlation function on the rising edge, and the late correlator samples the falling edge. Modern receivers sample the rising and falling edge of the peak simultaneously. Some early receivers used only one correlator and moved it back and forth between the rising and falling edge. This time-shared strategy is called a tau-dither loop. However, such a loop is not further considered in this book because it does not perform as well as strategies that sample both edges simultaneously.

$$r(t) = \sqrt{C} x(t - \hat{\tau}) + n(t)$$

**Figure 10.15** Delay lock loop (DLL) for white noise analysis of ranging precision. The function $n(t)$ is additive white noise (AWN).
A popular control strategy, called null seeking, attempts to equalize the early and late correlator values. In other words, a null seeking loop will shift the time of the replica code so that the difference between the early and late correlator values is zero. Such a loop is widely used because the difference in the correlator values is not sensitive to nuisance effects that may cause the absolute correlator values to change. For example, if the receiver moves under foliage, the absolute values will decrease, but this foliage attenuation will not be due to an authentic change in pseudorange. The null seeking strategy is robust to this nuisance effect because both correlator values will decrease and so their difference will remain approximately constant. Consequently, the null seeker will not confuse the signal attenuation with an effect that needs to be tracked.

The fixed time between the early and late correlator samples is called the correlator spacing. As Figure 10.16 shows, a wide correlator spacing is approximately one chip width or $d = 1$. The second spacing shown in the figure is around $d = 0.5$, and even narrower spacings are used. We shall discover that the correlator spacing is a powerful design parameter. Remember, $d$ is the correlator spacing measured in chips, and $dT_C$ is the corresponding time spacing.

The model in Figure 10.16 assumes that the received signal has been processed by the front end of the GPS receiver and is given by $\sqrt{C}x(t-\tau)+n(t)$, where $x(t)$ is the satellite code. $C$ is the power in the signal including the receiver’s antenna gain and any implementation loss. The signal amplitude is not $\sqrt{2C}$ because the signal does not include an RF carrier. In fact, our model is called a baseband model for the signal and receiver because it ignores the underlying carrier. We do not need the carrier to study the ranging prowess of spread spectrum signals. Do not worry—the carrier will reappear in Chapters 11 and 12.

Our input includes an additive disturbance, $n(t)$. As we know, GPS suffers from several such disturbances. These include natural radio noise, reflected signals, signals from the other GPS satellites, and other man-made interference. The arrival time of the satellite code, $\tau$, is
our unknown or estimand—we wish to accurately estimate this variable in spite of the strong additive disturbance, \( n(t) \).

The early and late samples shown in Figure 10.16 are denoted

\[
Z_E = S_E + N_E \\
Z_L = S_L + N_L
\]

where \( S_E \) and \( N_E \) are the signal and noise components of the early sample. \( S_L \) and \( N_L \) are the signal and noise contributions to the sample on the falling edge. The measurement noise contains the effects of natural noise, man-made interference, and multipath.

The signal components of the early and late samples are given by

\[
S_E = \frac{1}{T} \int_0^T \sqrt{C} x(t - \tau) x(t - (\hat{\tau} - dT_c/2)) \, dt \\
= \sqrt{C} R(\Delta \tau - dT_c/2) \\
S_L = \frac{1}{T} \int_0^T \sqrt{C} x(t - \tau) x(t - (\hat{\tau} + dT_c/2)) \, dt \\
= \sqrt{C} R(\Delta \tau + dT_c/2)
\]

Equation (10.25)

In this equation, \( \Delta \tau = \tau - \hat{\tau} \) is the error in the propagation time estimate, and \( R(\tau) \) is the auto-correlation function given in (9.82). \( T \) is the averaging time.

Equation (10.25) ignores the pre-correlation smoothing of the received signal due to front end filtering. The filters that precede the correlator tend to smooth or round the corners in the correlation peak, but our current discussion can neglect this complication. Thus we make the infinite pre-correlator bandwidth assumption.

Null tracking is enabled by subtracting the late sample from the early sample. The resulting difference is called the discriminator function, and is given by

\[
S_E - S_L
\]

**Figure 10.17** Discriminator functions for narrow and wide correlation peak sampling.
The signal component of this discriminator, $S_E - S_L$, is shown in Figure 10.17. Note that the shape of the discriminator is a function of the correlator spacing. The top trace is for $d = 1$ and the bottom trace is for $d = 0.5$. However, the slope of the discriminator near the origin is not affected by the spacing.

The estimated arrival time is the time at which the discriminator is zero. In the absence of $n(t)$, this zero crossing occurs at the true arrival time of the signal. However, disturbances push this zero crossing around, and this struggle is shown in Figure 10.18. As shown, the noise causes an error in the time estimate equal to 

$$
\Delta \tau = \frac{N_E - N_L}{\text{slope}(L_r)|_{\Delta \tau = 0}} \\
= \frac{\partial L_r}{\partial \Delta \tau|_{\Delta \tau = 0}} = \frac{\partial S_E}{\partial \Delta \tau|_{\Delta \tau = 0}} - \frac{\partial S_L}{\partial \Delta \tau|_{\Delta \tau = 0}} \\
= \frac{\sqrt{C}}{T_C} - \frac{-\sqrt{C}}{T_C} = \frac{2\sqrt{C}}{T_C} \\
\Delta \tau = \frac{T_C(N_E - N_L)}{2\sqrt{C}}
$$

This expression is valid for any additive disturbance, $n(t)$, with one proviso. The disturbance cannot be so large that it pushes the zero crossing out of the linear portion of the discriminator function shown in Figures 10.17 and 10.18.

The next section develops this equation for the more specific case where $n(t)$ is additive white noise (AWN) with a power spectral density equal to $N_0/2$ watts/hertz. We find that
white noise causes the time error, $\Delta \tau$, to be a random variable with zero mean and standard deviation equal to

$$\sigma_{\Delta \tau} = \sqrt{E[\Delta \tau^2]}$$

$$= \frac{T_C \sqrt{\text{var} \{ N_N - N_L \}}}{2 \sqrt{C}}$$

$$= T_C \sqrt{\frac{d}{4TC/N_0}} \text{ seconds}$$

$$= cT_C \sqrt{\frac{d}{4TC/N_0}} \text{ meters}$$

(10.28)

In this equation, $T$ is still the averaging time, and $c$ is the speed of light ($c \approx 3 \times 10^8$ m/s). When expressed in meters, this error is called the pseudorange error.

These equations may be enjoyed without suffering the proof provided in the next optional section. They reveal many important features of the GPS signal design. First, error decreases as the power in the signal, $C$, increases relative to the power spectral density of the noise, $N_0/2$. Indeed, the ratio, $C/N_0$ will play a major role in all that follows.

Equations (10.27) and (10.28) also reveal the utility of spread-spectrum signaling. With all other parameters held constant, ranging performance improves by reducing the chip width, $T_C$. Reducing the chip width increases the chipping rate, $1/T_C$, and the bandwidth. Thus ranging performance is improved by increasing the chipping rate—even when the signal power and noise power spectral density are held constant. Spread spectrum signaling provides a processing gain proportional to signal bandwidth when the goal is to provide precise ranging measurements. This processing gain also mitigates multipath, competing signals from other GPS satellites, and other man-made interference. In Section 10.7, we shall discuss the antmultipath properties of the GPS signal, but first we complete our noise analysis.

Figure 10.19 plots DLL error from (10.28) versus $C/N_0$ in dB-Hz. The averaging time, $T$, is 10 seconds, and the correlator spacing is equal to one chip width, $d = 1$. Three different chip widths are used. The curve for the P(Y)-code uses $T_C = 0.1$ microseconds, and the C/A-code curve assumes $T_C = 1$ microsecond. The final curve assumes that $T_C = 10$ microseconds. Hopefully, the virtue of spread spectrum signaling for ranging is clear.

Smaller values of correlator spacing improve DLL performance relative to what we see in Figure 10.19. Equation (10.28) suggests that we should make $d$ as small as possible, and many receiver manufacturers use $d = 0.1$. Values below 0.1 do not offer any additional benefit because the receiver includes filters that round the correlation peak shown in Figure 10.16. These filters are discussed in Chapter 11 and are needed to remove radio frequency interference (RFI) from other man-made radio systems that occupy adjacent frequency bands. They round the corners of the correlation peak and reduce the slope of $L_\tau$ for very small correlator spacings. From (10.27), we see that a reduction in slope increases the error.

The DLL filter shown in Figure 10.15 plays a major role in determining the performance of the delay lock loop because it sets the averaging time. Long averaging times are achieved with narrow bandwidths, $B_{\tau,1}$; short averaging times correspond to large $B_{\tau,1}$. In general, this averaging time must be chosen to balance two considerations. Longer averaging times atten-
ate the effect of noise, and we call this reduction integration gain. However, longer averaging times can be problematic if the receiver is moving. If the averaging time is too long, then the correlator will smear together measurements made at different locations and the dynamic performance will suffer. In other words, the loop bandwidth is chosen to balance noise performance against authentic signal dynamics. In Chapter 12, we will develop expressions for the pseudorange error that depend on the bandwidth of the loop filter, $B_{\tau,1}$, shown in Figure 10.15. This future analysis converts (10.28) into the following.

$$\sigma_{\Delta \tau} = cT_C \sqrt{\frac{dB_{\tau,1}}{2(C/N_0)}}$$ meters

As shown, the new expression replaces $1/2T$ with $B_{\tau,1}$. Such a replacement is reasonable because the bandwidth of a box car averager, like our correlator, is $B_{\tau,1} = 1/2T$. The longer we average, the narrower the bandwidth. We will take a closer look at the loop filter in Chapter 12.

### 10.6 Ranging Precision in the Presence of White Noise*

This optional section details the calculation of (10.28) above. For this analysis, $n(t)$ has a constant power spectral density $N_0/2$ watts/hertz for all GPS frequencies. This white noise model covers thermal noise in the receiver, natural noise received by the antenna, and any man-made noise that is broadband compared to the GPS signal. However, it is not appropriate for reflected signals or interference from signals that have a smaller bandwidth than the GPS receiver front end.

To conduct our noise analysis, we will need the Fourier transform pair developed in Section 8.6.
\[
S_n(f) = \mathcal{F}\{R_n(\tau)\}
\]
\[
R_n(\tau) = E[n(t)n(t-\tau)] = \mathcal{F}^{-1}\{S_n(f)\}
\]

(10.29)

For stationary processes, the power spectral density is the Fourier transform of the noise autocorrelation function. The autocorrelation function only depends on the time between the noise samples, \(\tau\), and is the inverse Fourier transform of the power spectral density.

For white noise, these become
\[
S_n(f) = \frac{N_0}{2}
\]
\[
R_n(\tau) = \mathcal{F}^{-1}\left\{\frac{N_0}{2}\delta(\tau)\right\} = \frac{N_0}{2}\delta(\tau)
\]

(10.30)

With these tools, our white noise analysis is fairly straightforward. Recall that the time error, \(\Delta \tau\), is equal to \(N_E - N_L\) divided by the slope of the discriminator function. In the present analysis, \(N_E\) and \(N_L\) are due to white noise and must be characterized by their mean and variance (or standard deviation). Correspondingly, the time (or pseudorange) error is also a random variable. The average pseudorange error is equal to zero because the noise samples have zero mean. However, individual values of \(N_E\) and \(N_L\) certainly deviate from zero. This fluctuation causes the ranging error to vary randomly. The variance of the pseudorange error is given by
\[
\sigma^2_{\Delta \tau} = E(\Delta \tau^2) = E\left\{\left(\frac{N_E - N_L}{\text{slope}(L_c)_{\Delta \tau=0}}\right)^2\right\}
\]
\[
= \frac{E\left\{N_E^2 - 2N_EN_L + N_L^2\right\}}{\left(\frac{2\sqrt{C}}{T_c}\right)^2}
\]

(10.31)

In pursuit of the numerator, we compute
\[
\text{var}\{N_E\} = \text{var}\{N_L\}
\]
\[
= E\{N_E^2\} = E\{N_L^2\}
\]
\[
E\{N_E^2\} = E\left\{\frac{1}{T} \int_0^T n(t) n(t-\tau - dT_c/2) dt \frac{1}{T} \int_0^T n(s) n(s-\tau - dT_c/2) ds\right\}
\]
\[
= \frac{1}{T^2} E\left\{\int_0^T \int_0^T n(t)n(s)x(t-\tau - dT_c/2)x(s-\tau - dT_c/2) dt ds\right\}
\]
\[
= \frac{1}{T^2} \int_0^T \int_0^T E\{n(t)n(s)\} x(t-\tau - dT_c/2)x(s-\tau - dT_c/2) dt ds
\]
\[
= \frac{1}{T^2} \int_0^T \int_0^T \delta(t-s) x(t-\tau - dT_c/2)x(s-\tau - dT_c/2) dt ds
\]

(10.32)
Next, we use the sifting property of the unit impulse function to provide

\[
E \{ N_E^2 \} = \frac{N_0}{2T^2} \int_0^T x(t - \tau - dT_C/2) x(t - \tau - dT_C/2) \, dt \\
= \frac{N_0}{2T^2} \int_0^T \tau^2 (t - \tau - dT_C/2) \, dt \\
= \frac{N_0}{2T} \tag{10.33}
\]

Noise is reduced by longer averaging times—a sensible result.

However, we still have not found any influence from the correlator spacing, \( d \). To this end, we compute the other term in the numerator of (10.31).

\[
E \{ N_E N_L \} = \frac{1}{T^2} E \left\{ \int_0^T n(t) x(t - \tau - dT_C/2) \, dt \int_0^T n(s) x(s - \tau + dT_C/2) \, ds \right\} \\
= \frac{1}{T^2} \int_0^T \int_0^T E \{ n(t) n(s) \} x(t - \tau - dT_C/2) x(s - \tau + dT_C/2) \, dt \, ds \\
= \frac{1}{T^2} \int_0^T \int_0^T \frac{N_0}{2} \delta(t - s) x(t - \tau - dT_C/2) x(s - \tau + dT_C/2) \, dt \, ds \\
= \frac{N_0}{2T^2} \int_0^T x(t - \tau - dT_C/2) x(t - \tau + dT_C/2) \, dt \\
= \frac{N_0}{2T} R(dT_C) \\
= \frac{N_0}{2T} (1 - d) \tag{10.34}
\]

Now finally, we can put the entire puzzle together.

\[
E \{ \Delta \tau^2 \} = \frac{E \{ N_E^2 - 2N_E N_L + N_L^2 \}}{(2\sqrt{C/T_C})^2} \\
= \frac{N_0}{T} - \frac{N_0}{T} (1 - d) \\
= \frac{dT_C^2}{4TC/N_0} \text{ seconds}^2 \\
\sigma_{\Delta \tau} = c T_C \sqrt{\frac{d}{4TC/N_0}} \text{ meters} \tag{10.35}
\]

This powerful result is discussed at the end of the last section, and we do not repeat that discussion here. Rather, we return only to the discussion of correlator spacing, \( d \). Decreasing spacing improves performance, but not because of any improvement in the discriminator. As discussed in the last section, the discriminator slope does not change with correlator spacing.
The error reduction exists because the two noise samples begin to cancel as the correlator spacing is reduced. After all, the samples are becoming more nearly simultaneous as the spacing is reduced, and so the early and late noise samples are becoming more correlated [Van Dierendonck et al. (1994)].

10.7 Ranging Precision in the Presence of Signal Reflections (Multipath)

As discussed in Section 5.2, multipath arises when multiple paths exist from the satellite to the user antenna. The primary path is usually a direct, unobstructed path from the satellite to the antenna, while the secondary paths usually include a reflection off a nearby object or the ground. These reflections confound the receiver by distorting the correlation peak. After all, our analysis so far has assumed that this peak is a pristine triangle. If additional signals arrive, they will contribute secondary peaks and the early and late correlator samples may not be centered on the true arrival time of the direct ray.

In general, the impact of multipath depends on:

- the amplitude of the reflected signal relative to the direct
- the delay of the reflected signal relative to the direct
- the phase of the reflected signal relative to the direct
- the rate of change of the relative phase

Let’s begin by exploring the phase relationship. The carriers for a direct ray and two reflected rays are shown in Figure 10.20. The top carrier is for a reflection that arrives in phase with the direct wave and therefore causes constructive interference—the direct ray is strengthened by the reflection. The bottom carrier arrives out of phase with the carrier. It causes destructive interference and weakens the direct ray. In this case, the signal is said to fade.

Figure 10.20 introduces vector diagrams for constructive and destructive interference. The length of each vector is the amplitude of the corresponding carrier. The angle between

![Figure 10.20](image-url)
two vectors gives the relative phase of the two carriers. The direct ray is shown as the dashed vector and has larger amplitude than either of the reflections. The solid vectors for the reflections are added to the direct ray. Constructive interference has the same angle, and so the length of the sum is greater than the length of the direct ray by itself. In contrast, the angle for destructive interference is 180°, and so the sum is shorter than the direct ray. In general, the phase of the reflection varies and assumes all possible angles relative to the direct ray. Hence, the true vector picture would show a reflection vector with angle and magnitude that continuously varies relative to the direct ray.

The reflected wave always arrives after the direct wave and creates a delayed correlation peak as shown in Figure 10.21. The ratio of the direct peak amplitude to the delayed peak amplitude is $\sqrt{C/P_M}$, where $C$ is the power in the direct signal and $P_M$ is the power in the reflected signal. The late peak will also be shifted in time by $\Delta \tau_M$ seconds, where $\Delta \tau_M$ is the delay of the multipath relative to the direct. If the interference is constructive, then the late peak will be added to the earlier peak. If the reflection interferes destructively, then the late peak is subtracted from the early peak.

The relative delay of the multipath, $\Delta \tau_M$, plays a major role in determining its effect. If the delay is long compared to a chip width, $T_C$, then the auto-correlation properties of the code will suppress the effect—this regime is explored in the next subsection. If the delay is smaller than a chip width, then narrow correlator spacings are helpful—this effect is quantified in Section 10.7.2. Finally, special antennas can be used to mitigate multipath provided the user is stationary. This technique is explored in Section 10.7.3.

### 10.7.1 Long-Delay Multipath

If the delay is long compared to a chip width (approximately 300 meters or 1 microsecond for the C/A-code), then multipath will not cause any pseudorange errors. This situation is detailed in Figure 10.22. No errors exist when the rising edge of the delayed peak does not touch the late correlator sample. This condition holds when
As shown, spread spectrum signaling may be helpful in multipath environments. As the chip width, $T_C$, is reduced, multipath vulnerability is reduced. If a wide correlator is used, $d = 1$, with the C/A-code ($T_C = 1.0$ microseconds), then errors are obviated for all differential path lengths over 450 meters. If the P(Y)-code is used ($T_C = 0.10$ microseconds), then the receiver can tolerate all differential path lengths greater than 45 meters.

Figure 10.22 and (10.36) reveal another virtue of narrow correlator spacing. The C/A-code user can reduce vulnerability by using a narrow correlator spacing. If a narrow correlator, $d = 0.1$, is used with the C/A-code, then the errors are obviated for all differential path lengths greater than 315 meters.

If the multipath delay is less than $T_C (1 + d/2)$, then the correlator pairs will usually move and pseudorange errors will result. As shown in Figure 10.22, constructive interference tends to move the correlation pair slightly to the right, and the measured pseudorange is longer than it should be. Destructive interference moves the correlation pair to the left. Although the reflected ray necessarily arrives after the direct ray, destructive interference causes the pseudorange to be measured short!

### 10.7.2 Short-Delay Multipath

Short-delay multipath has been analyzed by Braasch (1996), Van Dierendonck et al. (1992) and Enge (1999), and some typical results are shown in Figures 10.23 and 10.24. These unusual figures plot multipath error bounds. The top and bottom traces are the upper and lower bounds on multipath error for multipath that is 12 dB weaker than the direct ray, $C/P_M = 16$. 

\[
\Delta \tau_M - T_C \geq \frac{dT_C}{2}
\]
\[
\Delta \tau_M \geq T_C (1 + d/2)
\]
\[
\Delta \tau_M (d = 1) \geq 1.5T_C
\]
\[
\Delta \tau_M (d = 0.1) \geq 1.05T_C
\]

(10.36)
The upper trace is for constructive interference where the pseudorange is measured long and the error is positive. The lower trace is for destructive interference where the measured pseudorange is short. As the relative phase varies from $0^\circ$ (constructive interference) to $180^\circ$ (destructive interference), the multipath error swings between its upper and lower bounds. If this variation in phase is adequately rapid in time, then the multipath error can be attenuated using carrier smoothing—as described in Section 4.7.

If the differential path length is small, then the error bounds are independent of the correlator spacing. For these short delays, the error grows linearly with differential path length and grows with multipath amplitude. As the relative path length increases, the role of correlator spacing asserts itself. Smaller correlator spacings have two good effects. As shown in Figure

![Figure 10.23](image)

**Figure 10.23** Bounds on C/A-code pseudorange error due to multipath. The actual error varies between the indicated upper and lower bounds as the relative phase changes. The upper bound corresponds to constructive interference, and the lower bound corresponds to destructive interference. The amplitude of the multipath is 12 dB below the amplitude for the direct ray.

![Figure 10.24](image)

**Figure 10.24** Bounds on P(Y)-code and C/A-code pseudorange error due to multipath. The actual error varies between the indicated upper and lower bounds as the relative phase changes. The upper bound corresponds to constructive interference and the lower bound corresponds to destructive interference. The amplitude of the multipath is 12 dB below the amplitude for the direct ray.
10.23, they cause the error bounds to be smaller. As described by (10.36), they also cause the error bounds to decrease to zero for smaller delays.

Figure 10.24 compares the multipath error bounds for the C/A and P(Y)-codes. In both cases, the correlator spacing is one code chip width. So the correlator spacing for the C/A-code is one microsecond, and the spacing for the P(Y)-code is 0.1 microsecond. In the presence of multipath, the P-code receiver enjoys a significant advantage. However, Figure 10.24 also shows that much of this advantage can be recovered by using narrow correlators with the C/A-code.

Narrow correlators do require that the front end of the receiver have a wide bandwidth relative to the C/A-code chipping rate. In other words, the bandwidth of the circuits that precede the correlators must be large compared to the 1 Mcps chipping rate. Typical narrow correlator receivers have pre-correlator bandwidths of 16 MHz. Narrower bandwidths tend to round the corners of the correlation peak and mute the advantage of narrow spacings. On the other hand, pre-correlator bandwidths cannot be too wide. Otherwise, they will not attenuate signals from competing radio systems that operate in adjacent radio bands. This trade-off discourages the use of narrow correlators in conjunction with P(Y)-code receivers. Pre-correlator bandwidths several times greater than the P(Y) chipping rate would be unduly vulnerable to interference.

10.7.3 Multipath-Limiting Antennas

Two other techniques are available to combat multipath with a relatively short delay. If the relative phase between direct and reflected changes rapidly, then the receiver can simply average the pseudorange measurements. If the averaging time is long compared to the time required for the phase to change by half a wavelength, then the averaged multipath will be attenuated and the net effect will be small.

If the delay and phase rate are both small, then a special antenna may be helpful. The gain pattern for such a multipath-resistant antenna is shown in Figure 10.25 (dBSystems, 2000).
Contrast the gain pattern shown in Figure 10.25 to the gain pattern for a typical GPS patch antenna shown in Figure 10.6. As shown, the gain for the multipath-resistant antenna decreases much more rapidly when the satellite falls below 10°. Consider a satellite at low elevation (approximately 10°) and assume that the signal arrives on the direct path from the satellite, but is also reflected from below the antenna. GPS signals can, in fact, be reflected from below ground—they travel through the ground and then reflect off a layer of moist earth. After reflection, the multipath signal arrives at the antenna from an elevation angle that is the negative of the satellite elevation angle. If the satellite is at 10° elevation, then the reflected signal will arrive from an angle of −10°. Such reflections are troublesome at high-quality differential GPS stations. Moreover, receiver processing can do little to ameliorate this effect—the antenna alone must provide protection.

For the patch antenna, a reflection from −10° will be approximately 10 dB weaker than the direct signal from +10°. For the multipath-resistant antenna, a reflection from −10° will be approximately 30 dB weaker than the direct signal. This attenuation pays off with a much smaller multipath error in the code phase measurement. This promise is substantiated by Figure 10.26, which shows code minus carrier measurements for the multipath antenna on the left and for a standard patch antenna on the right. Over a short time period, code minus carrier measurements estimate the impact of multipath on code phase error because the code measurements are significantly more sensitive to multipath than the carrier measurements.

The multipath antenna characterized in Figure 10.25 has little gain above 35°. A second antenna is placed on top of the multipath-resistant antenna to fill in the response above 35°. Other multipath-rejecting antennas have been designed and some are based on a single antenna rather than a pair [Counselman (1999)].

10.8 Summary

The GPS signal is weak by the time it travels from orbit to the surface of the earth. The received power is proportional to $1/R^2$, where $R$ is the distance traveled. This attenuation is partially offset by the gain of the satellite transmit antenna. Indeed, this antenna focuses most of its transmitted power on the earth and the corresponding gain is a little greater than ten. How-
ever, most receiver antennas cannot employ such focusing. They have a hemispherical gain pattern, so they can receive GPS satellite signals coming from all skyward directions. Consequently, they do not amplify the received signals. All told, the received GPS signal power is approximately $10^{-16}$ watts.

In the front end of the GPS receiver, natural noise is much stronger than the GPS signal. Friis formula accounts for the noise that is both external to the receiver as well as the noise that is developed in the front end of the receiver. Happily, the GPS signal power becomes stronger than the competing noise after the correlation process de-spreads the GPS signal.

We analyzed GPS ranging performance in the presence of white noise. Ranging performance improves with $C/N_0$ and averaging time. It is also sensitive to the chipping rate of the spread spectrum codes. Faster codes are better, and narrow correlator spacings also help.

Signal reflections, known as multipath, are a nuisance. For long delays, reflections are attenuated by the spread spectrum codes, but short-delay multipath can be more troublesome. This scourge is treated by using narrow correlator spacings, averaging, special antennas and careful siting of the receiver antenna. Indeed, the antenna and proper siting can treat multipath problems that no amount of subsequent processing by the receiver will be able to cure.

### Homework Problems

10-1. GPS satellites direct their power toward the earth, but some power escapes around the edge of the earth and reaches geostationary satellites on the other side. What is the approximate GPS signal power at a geostationary satellite? Can this signal serve any purpose? What receiver processing could be done on the geostationary satellite to increase the utility of the GPS signal?

10-2. Derive the noise figure for a lossy element like a cable. (If desired, refer to Tsui (1995) or Vizmuller (1995) for some guidance.)

10-3. Compare the cascaded noise figures for the following two systems. System A is a cable followed by an amplifier followed by a receiver. System B uses the same elements, but in the following order: amplifier, cable and receiver. The cable has 5 dB of loss. The amplifier has a gain of 20 dB and a noise figure of 2 dB. The receiver has a gain of 50 dB and a noise figure of 7 dB.

10-4. Consider a cable followed by an amplifier. The cable loss is 1 dB for every 100 meters, and the amplifier noise figure is 7 dB. The source temperature is 290 K. How long can the cable be before the output signal-to-noise ratio is 5% of the input signal-to-noise ratio?

10-5. Calculate the error bounds due to multipath shown in Figures 10.23 and 10.24. (If desired, refer to Enge (1999) for some guidance.)

10-6. Calculate error bounds for the GPS carrier phase due to multipath. Comment on the relationship between these errors and the code phase errors.
References


dBSystems (2000), personal communication.


